International Journal of Theoretical Physics, Vol. 8, No. 6 (1973), pp. 443–450

Two Axisymmetric Black Holes cannot be in Static Equilibrium

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Received: 12 *February* 1973

Abstract

No static equilibrium configuration of two black holes can exist in an axisymmetric asymptotically flat vacuum space-time.

1. *Introduction*

In a previous paper one of the authors developed methods for dealing with a static axisymmetric space-time containing two bodies (Müller zum Hagen, 1970a, 1972). It was mentioned there that these methods can be used to disprove the existence of a static axisymmetric two black hole configuration; we shall prove this here.

A static axisymmetric‡ two black hole system which, by assumption, is

(A) asymptotically flat,

has roughly speaking the following properties:

(B) Each black hole acts as a body with *positive mass* because the potential V vanishes at the horizon only and tends to 1 at infinity.

(C) The two black holes can be *separated* by a 'plane'. This follows from the behaviour of the norms V, rV^{-1} of the static resp. axisymmetric Killing vectors:

  (a) $V = 0$ at the horizons only,

  (b) the gradient of r is nowhere vanishing (this is connected with the spherical topology of the horizons (Hawking, 1972)).

The separation property C plays the following role: As the gravitational field is attractive (due to B, A; cf. Müller zum Hagen, 1970c) two bodies

† Work partly supported by the Deutsche Forschungsgemeinschaft.
‡ For a definition of 'axisymmetry' see Carter (1972).

enclosed in two non-intersecting convex regions cannot remain in static equilibrium. This applies to our case as black holes have a 'convex' shape. Thus:

The properties A, B, C imply that two black holes cannot assume a static (axisymmetric) equilibrium configuration.

It is interesting to contrast the static two black *hole* problem with the static two *body* problem (Müller zum Hagen, 1972): The convex property C as well as the positivity property B are automatically fulfilled for a two black hole system. This is by no means true for general bodies: One can, for material bodies, construct static equilibrium configurations where A and B (or A and C) are fulfilled, but the third property, C (or B resp.), is violated.

We now give a brief outline of the proof:

First we shall prove the global existence of a Weyl coordinate system (Section 2), using arguments due to Carter (1970, 1972).

This will enable us to apply the methods of Müller zum Hagen (1970a, 1972) (Section 3): (i) we shall derive equilibrium conditions for a two black hole system; (ii) we shall obtain a contradiction by showing that those equilibrium conditions are not consistent with our assumptions. This is so because of the following properties of V:

(a) There exist equipotential surfaces K_1 and K_2, each enclosing one black hole only.
(b) On K_1 and K_2 the gradient of V points out into the exterior region (cf. B).
(c) K_1 and K_2 can be separated by a 'plane' (cf. C). This concept 'plane' will be made precise in the course of the proof (Theorem 3.1).

2. *The Global Weyl Coordinate System*

Assumptions

(A1) V^4 is a static, axisymmetric, simply connected solution of Einstein's vacuum field equations; in particular, V^4 is the metrical product of R^1 and a space-like simply connected hypersurface ^3M.
(A2) V^4 is asymptotically flat.
(A3) V^4 contains two black holes (for exact definitions of static black holes in terms of the interior structure of ^3M see Müller zum Hagen (1973)); no other incompletenesses occur in ^3M.

A more precise formulation of (A3) and (A2) may be given in the form: Any basis for the neighbourhoods of the black holes contains disconnected sets; but there is one basis consisting of neighbourhoods with at most two components. For an open neighbourhood U of the black holes one can find a compact set C of ^3M so that $^3M\backslash(C \cup U)$ is diffeomorphic to R^3

minus a compact set. In this region coordinates can be introduced, in which the norm V^2 of the static Killing vector and the 3-metric g_{ab} take the form:

$$V = 1 + c|x|^{-1} + O(|x|^{-2}) \qquad c \in R \tag{2.1}$$

$$g_{ab} = \delta + O(|x|^{-1}) \tag{2.2}$$

Lemma 2.1: The axis, i.e. the set of all degenerate group orbits, consists of three non-empty components: A_2 joins the two black holes, A_1 and A_3 join one black hole each with the infinite region.

Proof: Hawking (1972) has shown that black holes in ^3M must be topologically spheres. Hence the axisymmetric action on a horizon must have fixed points (end-points of an axis). If the system of axis and horizons is connected, one has the following order: infinity–axis–black hole–axis–black hole–axis–infinity. If the system were not connected, the orbit space of the axisymmetric static group would be multiply connected. But this contradicts the assumed simple connectedness of V^4.

Lemma 2.2: There exists a 2-surface ^2M orthogonal to the orbits of the axisymmetric static group G which meets any orbit of G exactly once. ^2M is uniquely defined up to a G-isometry of V^4.

Proof: (i) For any asymptotically flat axisymmetric stationary space-time the orbits admit locally orthogonal surfaces (Carter, 1969, 1970). Such a surface is locally uniquely defined by giving one point on it. Points of the axis can only occur as boundary points of ^2M.

(ii) A maximally extended orthogonal surface ^2M meets every orbit at least once. Otherwise the union of orbits met by ^2M would have a non-empty boundary, which obviously consists of full orbits. As the orbits in the static region are not null-surfaces, the local orthogonal surfaces to such a boundary orbit Z will cover a full neighbourhood of Z. Hence some $^2M'$ orthogonal to Z will meet ^2M, so it must coincide with ^2M on all orbits met by ^2M as well as by $^2M'$. Therefore $^2M \cup {}^2M'$ gives a proper extension of ^2M in contradiction to the assumed maximality of ^2M.

(iii) Generally, ^2M will meet every orbit several times (example below). But such a space V^4 will not be simply connected, as we can construct a non-trivial covering space by taking for every $x \in {}^2M$ the orbit through x and topologise the set of these orbits by using the locally 1–1-maps from the subsets of V^4 of orbits meeting a small neighbourhood of x. Hence, under assumption (A1), orbits are met only once.

Example: Consider $R^4(t, r, \varphi, z)$: $ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\varphi^2$. Remove $\{(r-2)^2 + z^2 < 1\}$ and identify (r, z, φ, t) and $(r, z, \varphi + \pi, t)$ on $\{(r-2)^2 + z^2 = 1\}$. The orthogonal surfaces are $\{\varphi = a\} \cup \{\varphi = a + \pi\}$ $(a \in R \bmod 2\pi)$, where the points on $\{r = 0\}$ are counted twice as boundary points. By a slight modification one gets a *smooth* example.

Corollary 2.1: ^2M is a manifold with boundary. The interior is homeomorphic to R^2 (since ^3M, hence ^2M is simply connected) and the boundary consists of three pieces of the axis.

Corollary 2.2: The metric of the space sections ^3M orthogonal to the static Killing vector can be written in the form:

$$ds^2 = g_{AB}\,dx^A\,dx^B + r^2(x^A)\,V^{-2}\,d\varphi^2 \qquad \varphi \in R \bmod 2\pi$$

where g_{AB} is the metric on ^2M.

Lemma 2.3: The function $r(x^A)$ has no critical points on ^2M, i.e. the gradient $r_{,A}$ vanishes nowhere.

Remark: This is a simple consequence of Morse's analysis of the relations between the critical points of functions and the underlying manifold (cf. Milnor, 1963; Morse & Heins, 1945). No theorem in these papers covers exactly our problem, since some work is concerned with non-degenerate critical points only (which we do not want to assume *a priori*) and other work is done under some assumptions which, in our case, are not fulfilled on the axis. For these reasons we shall give a direct proof.

Proof (by contradiction): As V^4 describes a static vacuum, r must be a real analytic function (Müller zum Hagen, 1970b); furthermore r is a non-trivial ($r \neq$ const.) solution of Laplace's equation $\Delta r := r_{,11} + r_{,22} = 0$ in isothermal coordinates $(g_{AB}dx^A dx^B \underset{*}{=} f^2(dx_1{}^2 + dx_2{}^2))$, which always exist locally (cf. Synge, 1964). Therefore any critical point p must be a saddle point of r, and the level set $L_{r_0} := \{x \in {}^2M \,|\, r(x) = r_0\}$ has a bifurcation in p (r_0 being the value of r at p).† From the asymptotic flatness it follows that one can find a curve γ in ^2M consisting of two arcs $\widehat{ab}$ and $\widehat{cb}$, where a and c are points on the axis segments A_1 and A_3 resp., r is monotonic on $\widehat{ab}$ and $\widehat{cb}$, and p is contained in the component S_0 of $^2M \backslash \gamma$ that does not contain the infinite region. Now from $\bar{S}_0$ we remove that component of L_{r_0} which contains p. The remaining set S is the sum of the following three sets:

S_1: The component of S which contains $\{r = 0\}$.

S_2: The union of the components of $S \backslash S_1$ which contain in any neighbourhood of p some points with $r < r_0$.

S_3: $S \backslash (S_1 \cup S_2)$.

Note that S_1 is non-empty and connected as a consequence of Lemma 2.1 and the fact that $r = 0$ on the horizon, cf. Carter (1972).

Case I: $S_2 = \varnothing = S_3$ is *impossible* as r is a continuous function on the simply connected set ^2M.

† Here and in the following we shall often use Hopf's principle (cf. Bochner & Yano, 1953): $\Delta\varphi = 0$ on a compact set C implies that the extremal values of φ will be assumed on $\dot{C}$ only.

Case II: $S_2 \neq \varnothing$. S_2 must contain a point q on γ with a value $r_1 < r_0$, otherwise there would be a minimum of r in the interior of S_2 (in ${}^2M$, $\bar{S}_2$ is compact; $\dot{S}_2 \subset \gamma \cup L_{r_0}$) in contradiction to $\Delta r = 0$. But, on the other hand, on both arcs $\widehat{aq}$ and $\widehat{cq}$ of γ, the set S_2 is separated from S_1 by L_{r_0}. Hence r could not be monotonic on $\widehat{ab}$ and $\widehat{cb}$.

Case III: $S_2 = \varnothing \neq S_3$. As p is a bifurcation point of L_{r_0} and a saddle point of r, a small connected neighbourhood U of p intersects S_1 in at least two disconnected parts, if S_2 vanishes. Two points q_1 and q_2 in such parts can be joined by an arc g_1 which lies entirely in S_1 (as S_1 is connected) and by a second arc g_2 lying in $U \cap (S_1 \cup \{p\})$. These arcs form a closed curve which separates ${}^2M$ into two parts, both containing entire components of S_3. One part, say S', must have compact closure. As r is not greater than r_0 on the boundary $g_1 \cup g_2$ the function r will take a maximum at an interior point of S'; again we have a contradiction to $\Delta r = 0$.

Lemma 2.4: The level sets $L_a := \{r = a\}$ are smooth lines homeomorphic to R^1 for every $a \in R^+$.

Proof: L_a cannot be empty as V^4 is asymptotically flat. Since r is an analytic function with no critical points (Lemma 2.3), each component of L_a is a closed smoothly embedded submanifold (cf. Müller zum Hagen *et al.*, 1973), which is homeomorphic either to the line R^1 or the circle T^1. As ${}^2M$ is simply connected, a component of some level set homeomorphic to T^1 would be the boundary of a compact subset in whose interior r must take an extremal value in contradiction to $\Delta r = 0$. The continuous extension of r onto the horizons exists and gives $r = 0$ on them (Carter, 1972), whence every component of L_a is a line running in both directions to infinity. From the asymptotic flatness it follows that r behaves monotonically at infinity, hence every L_a is connected.

Lemma 2.5: The metric on ${}^2M$ can be written as follows:

$$g_{AB}\, dx^A\, dx^B \underset{*}{=} f^2(dr^2 + dz^2) \qquad r \in R^+, z \in R$$

Proof: By Lemma 2.4, $r(x^1, x^2)$ possesses globally a conjugate harmonic function $z(x^1, x^2)$, defined uniquely up to a constant, which completes r to the complex analytic function $r + iz$ on ${}^2M$. z is strictly monotonic along the lines L_a. The other statements are simple consequences of (A2) and the preceding lemmas.

As an immediate consequence of the Lemmas 2.1–2.5, one obtains the following theorem by relabelling $(r, z) = (x_1, x_2)$:

Theorem 2.1: Under the assumptions (A1, 2, 3) the space time V^4 with the axis removed can be covered by a Weyl coordinate system:

$$\begin{aligned} ds^2 = {} & V^{-2}[e^{2U}(dx_1{}^2 + dx_2{}^2) + x_1{}^2\, dx_3{}^2] \\ & - V^2\, dt^2 \qquad x_1 \in R^+, x_2 \in R, x_3 \in R \bmod 2\pi, t \in R \end{aligned} \tag{2.3}$$

Remark: The coordinate system (2.3) gives a homeomorphism $(x_1, x_2, x_3) \to (r, z, \varphi)$ of ${}^3M$ onto $R \times (R^2 \setminus \{0,0\})$, where the latter set is represented (in the obvious way) in cylindrical coordinates (r, z, φ) with the axis $r = 0$ removed. Thereby (2.3) gives a natural (for our purposes) extension of ${}^3M$ to an ${}^3\bar{M} \cong R^3$: just fill in the axis. The functions x_1, x_2, V can be continuously extended onto ${}^3\dot{M} := {}^3\bar{M} \setminus {}^3M$ as follows: $x_1 = 0$ on ${}^3\dot{M}$; x_2 parametrises ${}^3\dot{M}$ by R, we can find five x_2-intervals on ${}^3\dot{M}$ so that the axis are: $A_1 =]-\infty, z_1[$, $A_2 =]z_2, z_3[$, $A_3 =]z_4, +\infty[$ and the black holes correspond to $B_1 := [z_1, z_2]$, $B_2 := [z_3, z_4]$; $V(x) = 0$, $x \in {}^3\dot{M} \Leftrightarrow x \in B_1 \cup B_2$.

3. *The Equilibrium Conditions*

Theorem 3.1: Under the assumptions (A1, 2, 3) there exist a coordinate plane $x_2 = b$ (in the Weyl coordinate system (2.3)) in ${}^3M$ which separates the two black holes in the following sense: B_1 and B_2 (defined in the remark to Theorem 2.1) have neighbourhoods U_1 and U_2 such that $x_2 < b$ on U_1 and $x_2 > b$ on U_2. V is constant on $\dot{U}_1 \cup \dot{U}_2$, the gradient $V_{,a}$ is a non-vanishing outgoing normal on $\dot{U}_1 \cup \dot{U}_2$, and $\dot{U}_1$ and $\dot{U}_2$ are homeomorphic to spheres.

Proof: (i) Let p be a point on the axis A_2 between B_1 and B_2 and b the x_2-value of p (in the sense of the remark to Theorem 2.1: $b \in]z_2, z_3[$). Then $x_2 < b$ is a neighbourhood of B_1 and $x_2 > b$ of B_2 respectively.

(ii) Assumption (the asymptotic behaviour) and the fact that $\{V = 0\} = B_1 \cup B_2$ imply that the sets $\{V < a;\ a \in]0,1[\}$ form a basis for the neighbourhoods of $B_1 \cup B_2$.

(iii) The non-critical values of V (the gradient of V vanishes nowhere on the level surface) are dense in $]0,1[$ as the critical values form a subset of measure zero (cf. Müller zum Hagen, 1970c).

(i), (ii), (iii) imply that we can find a non-critical value $c \in]0,1[$ so that $\{V = c\}$ contains two components K_i $(i = 1,2)$ such that:

(a) K_i is the boundary of a neighbourhood U_i of B_i.

(b) K_i does not intersect the set $\{x_2 = b\}$.

(c) On K_i the gradient $V_{,a}$ points out of U_i (remember: $V = 0$ on B_i and $V = 1$ at infinity).

Moreover, one has:

(d) K_i is homeomorphic to the sphere, because it is connected, invariant under the axisymmetry, and contains two points of the axis.

Theorem 3.2: There exists no space time V^4 which fulfills the assumptions (A1, 2, 3).

Proof: We divide the proof into three steps. In the first one we derive general equilibrium conditions, in the next step we specialise a certain surface of integration to our $K_i = \dot{U}_i$ constructed in Theorem 3.1. Finally we show that the equilibrium conditions lead to a contradiction.

Step I: The function U as defined in (2.3) can be continuously extended onto the axis $A_1 \cup A_2 \cup A_3$. U must vanish there, because the metric (2.3) is regular on the axis. In Synge (1964, p. 312) it has been shown that

$$U(x) = \int_\gamma v_A\, dx^A \qquad \text{where } \gamma \text{ is an arbitrary curve joining the axis with the point } x \tag{3.1}$$

where

$$W := \log V \qquad \text{and} \qquad (v_1, v_2) := (x_1[W_{,1}{}^2 - W_{,2}{}^2], 2x_1 W_{,1} W_{,2}) \tag{3.2}$$

This implies an equilibrium condition:

$$0 = \int_{\gamma_i} v_B\, dx^B, \qquad i = 1, 2 \qquad \text{where } \gamma_1 \text{ joins the segments } A_1 \text{ and } A_2 \text{ of the axis, and } \gamma_2 \text{ joins } A_2 \text{ and } A_3 \tag{3.3}$$

Introducing the flat metric $\hat{g}_{ab} dx^a dx^b :\underset{*}{=} dx_1{}^2 + dx_2{}^2 + x_1{}^2 dx_3{}^2$ on ${}^3M$, one can rewrite the equilibrium condition (3.3) and the essential field equation:

$$F_i := \int_{C_i} w_b\, d\hat{S}^b = 0; \qquad w_b := W_{,2} W_{,b} - \tfrac{1}{2}\hat{g}_{2b} W_{,c} W_{,d}\, \hat{g}^{cd} \tag{3.4}$$

$$\hat{\Delta} W := \hat{\nabla}^a W_{,a} = 0 \qquad \text{on } {}^3M \tag{3.5}$$

The quantities and operators with '^' are defined with respect to $\hat{g}_{ab}$; the surface C_i is obtained by rotating the curve γ_i with the axisymmetric group.

Step II: Now we choose the K_i as constructed in the proof of Theorem 3.1 as the integration surfaces C_i. As dS^a is parallel to the gradient of V, hence of W, (3.4) leads to:

$$0 = \int_{K_i} (w_b, W_{,a}\hat{g}^{ab})(W_{,c} W_{,d}\hat{g}^{cd})^{-1/2}\, d\hat{S} \tag{3.6}$$

Due to the fact that $W = \text{const}$ on K_i and to the asymptotic behaviour (2.1, 2.2) we obtain for a solution of a Laplace equation (3.5) the following integral representation:

$$W(x) = \sum_{i=1}^{2} \int_{K_i} \rho(x, \tilde{x})^{-1} \sigma(\tilde{x})\, d\hat{\tilde{S}} \tag{3.7}$$

Here $\rho(x, \tilde{x})$ is the Euclidean distance between x and $\tilde{x}$; $\tilde{x}$ is a point in the surface element $d\hat{\tilde{S}}$, and $\sigma := W_{,a} n^a$ is the product of $W_{,a}$ with the outer unit normal n^a of K_i. Inserting (3.7) into (3.6), we obtain:

$$0 = \int_{K_1} \left[\int_{K_2} (x_2 - \tilde{x}_2)\, \rho^{-3} \sigma(\tilde{x})\, d\hat{\tilde{S}} \right] \sigma(x)\, d\hat{S} \tag{3.8}$$

Step III: The integrand in (3.8) is strictly positive because:

(i) $(x_2 - \tilde{x}_2) > 0$ (convexity; Theorem 3.1)
(ii) $\sigma > 0$ ($V_{,a}$ points outward; Theorem 3.1) q.e.d.

Since the extension of the proof to the cases of more black holes is obvious, we have:

Corollary: Two or more axisymmetric black holes cannot exist in a static equilibrium in an asymptotically flat vacuum space.

Finally, let us remark that if W were the potential (fulfilling $\hat{\Delta} W = 0$) of Newton's gravitational theory, then the quantity F_i of (3.4) would be precisely the x_2-component of the gravitational force acting on the volume enclosed by C_i. This can be seen from the fact that w_b is a part of the stress tensor of the gravitational field:

$$W_{ab} := W_{,a} W_{,b} - \tfrac{1}{2}\hat{g}_{ab} W_{,c} W_{,d} \hat{g}^{cd}$$

and (having used Stoke's theorem) from:

$$F_i \underset{*}{=} \int_{C_i} W_{2b}\, d\hat{S}^b \underset{*}{=} \int W_{,2} \hat{\Delta} W \, d\hat{V}$$

Consequently the right-hand side of (3.8) is the total force between two surface layers K_1 and K_2 with surface density σ.

Acknowledgement

This work is based on a talk of one of the authors at the "Conference On Relativity and Related Topics", Brussels 1971.

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